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Population Projection upto the Lower Limit of the Reproductive Period

THE size of the population under stable or stationary conditions is of considerable interest to planners and has recently been the topic of many investigations (Keyfitz, 1975; Singh *et al,* 1981). The population that is governed by a regime of unchanging fertility and mortality schedule for a long time (with no migration) is called the stable population. The age structure of the population remains fixed and the size changes with a constant rate of increase, which depends on fertility and mortality schedules of the population concerned. If the fertility and mortality schedules are such that the rate of increase becomes zero, then the size of the population also becomes fixed and such a population is known as stationary population.

The size of the population and its growth depends on many characteristics of the population as well as on the manner of reduction in the schedules of fertility and/or mortality. Making an abrupt change in maternity function $p(a)m(a)$ (defined in the next section), an expression to find the size of population at any time t for $t \Leftarrow$ alfa, where alfa is the lower limit of the reproductive period, is worked out. For the simplicity, expressions deal with female sex only. The model is derived below; it is then illustrated by taking some numerical values of the parameters involved in the model.

Population Size at Time *t* for *t* < *alfa*

A formula to find the size of population at any time t for $t \ll$ alfa is derived here. For this purpose, we need the total number of births occurring in (0, t) for $t <$ alfa. An expression for the birth trajectory is given by Frauenthal (1975) and for completeness, it is given below :

Lotka (1939) has given a renewal equation for female births $B(t)$, at time t for a population which is closed to migration, as follows:

$$B(t) = \int_{t}^{\infty} l(a-t)\,\frac{p(a)}{p(a-t)}\,m(a, t)da$$

$$+ \int_{0}^{t} B(t-a)\,p(a)\,m(a, t)da \qquad (1)$$

$$\text{for } t > 0$$

where $l(a)$ is the age distribution at time $t = 0$, $p(a)$ is the fraction of female population that survives to age a, $m(a)da$ is the probability that a female who is of age a will bear a female child in next da period of her life.
If the base line population is stable then, we have

$$l(a) = B(0)\,e^{-ra}p(a) \qquad (2)$$

where $B(0)$ is the instantaneous birth rate at $t = 0$. Thus from equations (1) and (2), we get

$$B(t) = B(0)\,e^{rt} \int_{t}^{\infty} e^{-rt}\,p(a)\,m(a, t)da$$

$$+ \int_{0}^{t} B(t-a)\,p(a)\,m(a, t)da \qquad (3)$$

$$\text{for } t > 0$$

For $t \leqslant \alpha$, equation (3) reduces to

$$B(t) = B(0)\,e^{rt} \int_{t}^{\infty} e^{-ra}\,p(a)\,m(a, t)da \qquad (4)$$

If an abrupt change in maternity function at time zero is allowed such that

$$p(a)\,m(a, t) = \delta p(a)\,m(a) \qquad (5)$$

where δ is a constant of reduction in the maternity function.
Now equation (4) becomes

$$B(t) = \delta\,B(0)\,e^{rt} \qquad (6)$$

because $\int_{t}^{\infty} e^{-ra}\,p(a)\,m(a)\,da = 1$ for $t \leqslant \alpha$ is the characteristic equation for the intrinsic rate of increase γ. The equation (6) gives the birth trajectory at any time $t(t \leqslant \alpha)$ obtained by Frauenthal (1975).

Now the population at any time t is the sum of two populations, viz., (i) the survivors of the birth in $(0, t)$ upto time t and (ii) the survivors of the initial population upto time t.

Let us consider part (i): The total number of births at time y; $0 \leqslant y \leqslant t < \alpha$ is given by

$$B(0)\,\delta e^{-ry} \tag{7}$$

and the probability that they will survive upto time t is $p(t - y)$. Thus, the total survivors upto time t of births in $(0, t)$ are

$$\int_0^t B(0)\,\delta e^{-ry}\,p(t - y)dy \tag{8}$$

on putting $t - y = Z$, we have the equation (8) as

$$B(0)\,\delta e^{rt} \int_0^t e^{-rz} p(Z)dZ. \tag{9}$$

Let us consider part (ii): The population at age y in the base line population, which is initially stable, is

$$B(0)\,e^{-ry}p(y) \tag{10}$$

and the chance that they will survive upto time t is $p(y + t)/p(y)$ and hence the total survivors of the initial population at time t are

$$\int_0^w B(0)\,e^{-ry}\,p(y)\,\frac{p(y+t)}{p(y)}\,dy$$

i.e. $$\int_0^w B(0)\,e^{-ry}\,p(y + t)\,dy \tag{11}$$

where w is the highest age of life.

On putting $y + t = Z$, we have the equation (11) as

$$B(0)\,e^{rt} \int_t^w e^{-rz}\,p(Z)\,dZ. \tag{12}$$

Hence the population at time t can be obtained by summing the equations (9) and (12) as

$$B(0)\,\delta\,e^{rt} \int_0^t e^{-rz}\,p(Z)\;dZ + B(0)\,e^{rt} \int_t^w e^{-rz}\,p(Z)\,dZ. \tag{13}$$

By simplifying and putting

$$\int_0^w B(0)\, e^{-rz}\, p(Z)\, dZ = 1$$

The equation (13) becomes

$$e^{rt} - e^{rt}\,(1 - \delta) \int_0^t B(0)\, e^{-rz}\, p(Z)\, dZ$$

$$\text{i.e. } [1 - (1 - \delta)\bar{A}_t]\, e^{rt} \qquad \text{for } t \leqslant \alpha \tag{14}$$

where

$$\bar{A}_t = \int_0^t B(0)\, e^{-rz}\, p(Z)\, dZ$$

is the proportion of population upto age t in the initial population. Thus, using the formula (14), one can get the size of population upto the time t for $t \leqslant \alpha$, the lower limit of the reproductive period. The male population can be obtained from:

$$\text{Male population} = \text{female population} \times \text{sex ratio at birth} \times \frac{\text{life expectancy for males}}{\text{life expectancy for females}}$$

Illustrations

The population projection model as obtained by equation (14) is illustrated for different values of $t \leqslant \alpha$ taking numerical values of the other parameters.

For example, if we take $r = 0.0212$ and $R_0 = 1.6$ as India's population growth rate and net reproduction rate respectively, $\bar{A}_t = 0.167433$ for $t = 5$ (this value is taken from the Regional Model Life Table of Coale and Demeny 1966, see South Model, level 13). This level is chosen because of similarity with the India's mortality experiences and $\delta = R_0^*/R_0$ where R_0^* is the desired value of net reproduction rate obtained after making reduction in the maternity function. If the value of $R_0^* = 1$, then the population size after 5 years would be 1.1002 times of the initial population. The population sizes for different values of $t \leqslant \alpha$ and R_0^* are given in Table 1.

TABLE 1—**POPULATION** SIZES FOR DIFFERENT VALUES OF $t < x$ and R_0^*

Time (t)	Population Sizes* far		
	$R^*_0 = 1.5$	$R^*_0 = 1.3$	$R^*_0 = 1$
5	1.1002	1.0769	1.0420
10	1.2147	1.1719	1.1077
15	1.3409	1.2715	1.1739

***Assuming initial** population size **to** be 1.

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